

ANOTHER GENERALIZATION OF A THEOREM OF BAKER AND DAVENPORT

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ABSTRACT. Dujella and Pethő, generalizing a result of Baker and Davenport, proved that the set $\{1, 3\}$ cannot be extended to a Diophantine quintuple. As a consequence of our main result, it is shown that the Diophantine pair $\{1, b\}$ cannot be extended to a Diophantine quintuple if $b - 1$ is a prime.

1. INTRODUCTION

A set of m distinct positive integers $\{a_1, \dots, a_m\}$ is called a *Diophantine m -tuple* if $a_i a_j + 1$ is a perfect square. Diophantus studied sets of positive rational numbers with the same property, particularly he found the set of four positive rational numbers $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$. But the first Diophantine quadruple was found by Fermat observing that the set $\{1, 3, 8, 120\}$ is a Diophantine quadruple. Moreover, Baker and Davenport, in their classical paper [1], proved that the set $\{1, 3, 8\}$ cannot be extended to a Diophantine quintuple.

In 1997, Dujella [3] obtained that the Diophantine triples of the form $\{k - 1, k + 1, 4k\}$ cannot be extended to a Diophantine quintuple for $k \geq 2$. The Baker-Davenport's result corresponds to $k = 2$. In 1998, Dujella and Pethő [6] proved that the Diophantine pair $\{1, 3\}$ cannot be extended to a Diophantine quintuple. In 2008, Fujita [7] obtained a more general result by showing that the Diophantine pair $\{k - 1, k + 1\}$ cannot be extended to a Diophantine quintuple for any integer $k \geq 2$. In 2004, Dujella [5] proved that there are only finitely many Diophantine quintuples. A folklore conjecture states that there does not exist a Diophantine quintuple, however, its proof

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seems well beyond the current techniques. Let

$$d_+ = d = a + b + c + 2abc + 2\sqrt{(ab+1)(ac+1)(bc+1)}.$$

A stronger version of this conjecture is the following

Conjecture. *If $\{a, b, c, d\}$ is a Diophantine quadruple and $d > \max\{a, b, c\}$, then $d = d_+$.*

We introduce the concept of *regular quadruple*. A Diophantine quadruple $\{a, b, c, d\}$ is regular if and only if $(a + b - c - d)^2 = 4(ab + 1)(cd + 1)$. Therefore, the quadruples in the above conjecture are regular.

The aim of this paper is to consider the extensibility of the Diophantine pair $\{1, b\}$ and to give a new generalization of the mentioned theorems by Baker and Davenport [1] and Dujella and Pethő [6].

We remark that there exists a related result for higher power Diophantine pairs. Bennett [2] proved that the pairs $\{1, b\}$ cannot add a positive integer c such that $b+1, c+1, bc+1$ are both k th power, for any integer $k \geq 3$. For other results concerning Diophantine m -tuples and their generalizations we refer the interested reader to the homepage <http://web.math.pmf.unizg.hr/~duje/dn.html>.

If $\{1, b, c\}$ is a Diophantine triple, then there exists positive integers r, s and t such that

$$b+1 = r^2, \quad c+1 = s^2, \quad bc+1 = t^2.$$

Thus we have

$$(1) \quad t^2 - bs^2 = 1 - b.$$

Usually, we cannot get all pairs (s, t) solutions of Pell equation (1) without a condition on the parameter b . However, when $b-1$ is a prime power, the solutions (s, t) to equation (1) are easy to be parameterized by b . In fact, we know that there are at most $2^{\omega(l)}$ (here $\omega(l)$ denotes the number of distinct prime factors of l) classes solutions to the Pell equation $x^2 - Dy^2 = l$ with $(x, y) = 1$. This leads to confirm that all positive solutions to equation (1) can be expressed as

$$(2) \quad t + s\sqrt{b} = (\pm 1 + \sqrt{b})(r + \sqrt{b})^k = (\pm 1 + \sqrt{b})(T_k + U_k\sqrt{b}), \quad k \geq 0,$$

where (T_k, U_k) is the k th nonnegative integer solutions to the Pell equation $T^2 - bU^2 = T^2 - (r^2 - 1)U^2 = 1$. One can show that

$$(3) \quad T_0 = 1, \quad T_1 = r, \quad T_{k+2} = 2rT_{k+1} - T_k, \quad k \geq 0$$

$$(4) \quad U_0 = 0, \quad U_1 = 1, \quad U_{k+2} = 2rU_{k+1} - U_k, \quad k \geq 0.$$

Thus, we have

$$(5) \quad (s, t) = (s_k^{(\pm)}, t_k^{(\pm)}) = (T_k \pm U_k, \pm T_k + bU_k)$$

and

$$(6) \quad c_k^{(\pm)} = c = s^2 - 1 = T_k^2 \pm 2T_k U_k + U_k^2 - 1 = r^2 U_k^2 \pm 2T_k U_k.$$

We study the extensibility of the Diophantine triples

$$\{1, b, c_k^{(\pm)}\}, \quad \text{for } k = 1, 2, \dots$$

and prove the following

Theorem. *If $\{1, b, c_k^{(\pm)}, d\}$ is a Diophantine quadruple with $d > c_k^{(\pm)}$, then $d = c_{k+1}^{(\pm)}$.*

One can check that the Diophantine quadruple $\{1, b, c_k^{(\pm)}, c_{k+1}^{(\pm)}\}$ is regular.

If $b - 1$ is a prime then we recall that the corresponding generalized Pell equation possesses at most two classes of solutions. Thus, the next result is a straightforward consequence of Theorem.

Corollary. *If $b - 1$ is a prime, then the pair $\{1, b\}$ cannot be extended to a Diophantine quintuple. Moreover, any Diophantine quadruple which contains the pair $\{1, b\}$ is regular.*

By Proposition 4.6 of [8], if $c > \max\{100b^7, 20b^8\}$, then $d = d_+$. By (2), if $k \geq 8$, we get

$$s \geq \frac{1}{2\sqrt{b}}(\sqrt{b} - 1)(2\sqrt{b})^8 > 2^6 b^4$$

and $c = s^2 - 1 > 100b^8$. Consequently, we only need to consider $1 \leq k \leq 7$. There are 14 cases:

$$c_1^{(\pm)} = r^2 \pm 2r,$$

$$c_2^{(\pm)} = 4r^4 \pm (8r^3 - 4r),$$

$$c_3^{(\pm)} = 16r^6 - 8r^4 + r^2 \pm (32r^5 - 32r^3 + 6r),$$

$$c_4^{(\pm)} = 64r^8 - 64r^6 + 16r^4 \pm (128r^7 - 192r^5 + 80r^3 - 8r),$$

$$c_5^{(\pm)} = 256r^{10} - 384r^8 + 176r^6 - 24r^4 + r^2 \pm (512r^9 - 1024r^7 + 672r^5 - 160r^3 + 10r),$$

$$c_6^{(\pm)} = 1024r^{12} - 2048r^{10} + 1408r^8 - 384r^6 + 36r^4$$

$$\pm(2048r^{11} - 5120r^9 + 4608r^7 - 1792r^5 + 280r^3 - 12r),$$

$$c_7^{(\pm)} = 4096r^{14} - 10240r^{12} + 9472r^{10} - 3968r^8 + 736r^6 - 48r^4 + r^2$$

$$\pm(8192r^{13} - 24576r^{11} + 28160r^9 - 15360r^7 + 4032r^5 - 448r^3 + 14r).$$

Notice that $1 < b < c_k^{(\pm)}$ except for $1 < c_1^{(-)} < b$. We set $r' = r - 1$ in the case $c = c_1^{(-)}$. Then, the triple $\{1, c, b\}$ is $\{1, r'^2 - 1, r'^2 + 2r'\}$. It is similar to $\{1, b, c_1^{(+)}\}$.

2. THE SYSTEM OF PELL EQUATIONS

Let us consider a Diophantine triple $\{1, b, c\}$. In order to extend this triple to a Diophantine quadruple $\{1, b, c, d\}$, we have to solve the system

$$d + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2,$$

in integers x, y, z . Eliminating d , we obtain the following system of Pell equations

$$(7) \quad z^2 - cx^2 = 1 - c,$$

$$(8) \quad bz^2 - cy^2 = b - c,$$

$$(9) \quad y^2 - bx^2 = 1 - b.$$

By [4, Lemma 1], if $(z_0, x_0), (z_1, y_1)$ and (y_2, x_2) are the minimal solutions of (7), (8) and (9), respectively, then all solutions of (7), (8) and (9) are given by

$$(10) \quad z + x\sqrt{c} = (z_0 + x_0\sqrt{c})(s + \sqrt{c})^m, \quad m \geq 0,$$

$$(11) \quad z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n, \quad n \geq 0,$$

$$(12) \quad y + x\sqrt{b} = (y_2 + x_2\sqrt{b})(r + \sqrt{b})^l, \quad l \geq 0.$$

Using the above expressions, refer to [4, Lemma 1], it is easy to get the following corresponding sequences to solutions. All solutions of (10) are given by $z = V_m$ for some integer $m \geq 0$, where

$$V_0 = z_0, \quad V_1 = sz_0 + cx_0, \quad V_{m+2} = 2sV_{m+1} - V_m,$$

and all solutions of (11) are given by $z = W_n$ for some integer $n \geq 0$, where

$$W_0 = z_1, \quad W_1 = tz_1 + cy_1, \quad W_{n+2} = 2tW_{n+1} - W_n.$$

From the sequence $\{W_n\}$ the corresponding solutions of (11) are also given by $y = A_n$, $n \geq 0$, where

$$(13) \quad A_0 = y_1, \quad A_1 = ty_1 + bz_1, \quad A_{n+2} = 2tA_{n+1} - A_n.$$

From (12), we conclude that $y = B_l$, for some integer $l \geq 0$, where

$$(14) \quad B_0 = y_2, \quad B_1 = ry_2 + bx_2, \quad B_{l+2} = 2rB_{l+1} - B_l.$$

Similar to [7, Lemma 5], we have the following result.

Lemma 1. *Assume that $\{1, b, c', c\}$ is not a Diophantine quadruple for any c' with $0 < c' < c_{k-1}^{(\pm)}$. Then, neither $V_{2m+1} = W_{2n}$ nor $V_{2m} = W_{2n+1}$. Moreover,*

(1) *If $V_{2m} = W_{2n}$ has a solution, then we have $z_0 = z_1 = \pm 1$.*

(2) *If $V_{2m+1} = W_{2n+1}$ has a solution, then we have $z_0 = \pm t$, $z_1 = \pm s$ and $z_0 z_1 > 0$.*

Using Lemma 1, we obtain

Lemma 2. *Assume that $\{1, b, c', c\}$ is not a Diophantine quadruple for any c' with $0 < c' < c_{k-1}^{(\pm)}$, for $b \geq 8$. $A_{2n} = B_{2l+1}$ has no solution. Moreover, if $A_{2n} = b_{2l}$ then $y_2 = 1$. In other cases, we have $y_2 = \pm 1$.*

Proof. By induction on (13) and (14), we easily get

$$\begin{aligned} A_{2n} &\equiv y_1 \pmod{b}, & A_{2n+1} &\equiv ty_1 \pmod{b}, \\ B_{2l} &\equiv y_2 \pmod{b}, & B_{2l+1} &\equiv ry_2 \pmod{b}. \end{aligned}$$

From [5, Lemma 1] we have

$$1 \leq |y_2| \leq \sqrt{\frac{(r-1)(b-1)}{2}} < 0.8b^{3/4} < 0.5b,$$

for $b \geq 8$.

If $A_{2n} = B_{2l}$ or B_{2l+1} , then by the case (1) of Lemma 1, we get $y_1 = 1$ as $z_1 = \pm 1$ and $bz_1^2 - cy_1^2 = b - c$, $y_1 \geq 1$.

- $A_{2n} = B_{2l}$. We have $1 \equiv y_2 \pmod{b}$. Since $|y_2| < b^{3/4}$, then $y_2 = 1$.
- $A_{2n} = B_{2l+1}$. We have $1 \equiv ry_2 \pmod{b}$, and so $r \equiv r^2y_2 \pmod{b}$. As $b+1 = r^2$, then we get $y_2 \equiv r \pmod{b}$. Since $|y_2| < 0.5b$ and $r < 0.5b$ for $b \geq 8$, thus $y_2 = r$. By (12), $x_2^2 = (y_2^2 + b - 1)/b = (r^2 + b - 1)/b = 2$. It is impossible.

If $A_{2n+1} = B_{2l}$ or B_{2l+1} , the case (2) of Lemma 1 helps to get $y_1 = r$ as $z_1 = \pm s$ and $bz_1^2 - cy_1^2 = b - c$, $y_1 \geq 1$.

- $A_{2n+1} = B_{2l}$. This implies $rt \equiv y_2 \pmod{b}$. By (5), $t = \pm T_k + bU_k \equiv \pm T_k \pmod{b}$. One can check that $T_k \equiv 1$ or $r \pmod{b}$ by $T_0 = 0$, $T_1 = r$, $T_{k+2} = 2rT_{k+1} - T_k$. So we have $y_2 \equiv \pm r$ or $\pm 1 \pmod{b}$. The case $y_2 \equiv \pm r \pmod{b}$ gives a contradiction like in case (2). Then we have $y_2 \equiv \pm 1 \pmod{b}$. It results $y_2 = \pm 1$.

- $A_{2n+1} = B_{2l+1}$. In the case, we have $rt \equiv ry_2 \pmod{b}$. With $\gcd(r, b) = 1$, we deduce $y_2 \equiv t \equiv \pm 1 \pmod{b}$ so $y_2 = \pm 1$ again. \square

We will determine the integer solutions (x, y, z) of system

$$(15) \quad \begin{cases} y^2 - bx^2 = 1 - b, \\ z^2 - cx^2 = 1 - c. \end{cases}$$

From the above result, we have to solve the equation

$$(16) \quad x = P_l = Q_m,$$

where

$$(17) \quad P_l = \frac{1}{2\sqrt{b}} \left((y_2 + x_2\sqrt{b})\alpha^l - (y_2 - x_2\sqrt{b})\alpha^{-l} \right),$$

$$(18) \quad Q_m = \frac{1}{2\sqrt{c}} \left((z_0 + x_0\sqrt{c})\beta^m - (z_0 - x_0\sqrt{c})\beta^{-m} \right),$$

and $\alpha = r + \sqrt{b}$ and $\beta = s + \sqrt{c}$ are solutions of Pell equations $T^2 - bU^2 = 1$ and $W^2 - cV^2 = 1$, respectively. Considering all solutions $x = P_l$ of the equation $y^2 - bx^2 = 1 - b$, we have

$$(19) \quad P_0 = x_2, \quad P_1 = rx_2 + y_2, \quad P_{l+2} = 2rP_{l+1} - P_l.$$

All solutions $x = Q_m$ of the equation $z^2 - cx^2 = 1 - c$ are determined by

$$(20) \quad Q_0 = x_0, \quad Q_1 = sx_0 + z_0, \quad Q_{m+2} = 2sQ_{m+1} - Q_m.$$

Referring to Lemma 2, there are two types of fundamental solutions as follows:

Type I: $l \equiv m \equiv 0 \pmod{2}$, $x_0 = x_2 = 1$, $z_0 = \pm 1$, $y_2 = 1$.

Type II: $m \equiv 1 \pmod{2}$, $x_0 = r$, $x_2 = 1$, $z_0 = \lambda_1 t$, $y_2 = \lambda_2$, $\lambda_1, \lambda_2 \in \{-1, 1\}$.

3. GAP PRINCIPLE

We will consider the following linear form in logarithms

$$(21) \quad \Lambda = l \log \alpha - m \log \beta + \log \gamma,$$

where

$$\gamma = \frac{\sqrt{c}(y_2 + x_2\sqrt{b})}{\sqrt{b}(z_0 + x_0\sqrt{c})}.$$

Lemma 3. *If $P_l = Q_m$ has solution (l, m) with $m \geq 1$, then $0 < \Lambda < \beta^{-2m}$.*

Proof. Put

$$E = \frac{y_2 + x_2\sqrt{b}}{\sqrt{b}}\alpha^l \quad \text{and} \quad F = \frac{z_0 + x_0\sqrt{c}}{\sqrt{c}}\beta^m.$$

It is clear that $E, F > 1$ if $m \geq 1$. Then equation $P_l = Q_m$ becomes

$$E + \frac{b-1}{b}E^{-1} = F + \frac{c-1}{c}F^{-1}.$$

Since $c > b \geq 8$, we have $\frac{c-1}{c} > \frac{b-1}{b}$. It follows that

$$(22) \quad E + \frac{b-1}{b}E^{-1} > F + \frac{b-1}{b}F^{-1},$$

and hence

$$(E - F)(EF - \frac{b-1}{b}) > 0.$$

So we get $E > F$. Moreover, by (22) we have

$$0 < E - F < \frac{c-1}{c}E^{-1} < E^{-1} < F^{-1}.$$

Therefore, we have $\Lambda > 0$ and

$$\Lambda = \log \frac{E}{F} = \log \left(1 + \frac{E-F}{F} \right) < \frac{E-F}{F} < F^{-2}.$$

Considering all cases in types I, II, we have $\Lambda < \beta^{-2m}$. \square

Put

$$(23) \quad \lambda = \begin{cases} 0, & \text{if the solution } (l, m) \text{ is of Type I,} \\ 1, & \text{if the solution } (l, m) \text{ is of Type II with } \lambda_1 = 1, \\ -1, & \text{if the solution } (l, m) \text{ is of Type II with } \lambda_1 = -1. \end{cases}$$

We obtain the following result.

Lemma 4. *If $P_l = Q_m$ has a solution (l, m) with $m \geq 1$, then for $r \geq 1000$, we have*

$$(24) \quad |(l - \lambda) \log \alpha - m \log \beta| < \frac{2}{\sqrt{b}}.$$

Proof. By the definition of Λ in (21), we have

$$(25) \quad |(l - \lambda) \log \alpha - m \log \beta| = |\Lambda - \log \gamma - \lambda \log \alpha| \leq |\Lambda| + |\log \gamma + \lambda \log \alpha|.$$

One can easily get $0 < \Lambda < \frac{1}{4c}$. To estimate inequality (25), we will consider three cases according to the values of λ .

Case I: If $\lambda = 0$, then the solution (l, m) is of Type I. It implies $x_0 = x_2 = 1$, $z_0 = \pm 1$, $y_2 = 1$. We have

$$\gamma = \frac{\sqrt{c}(y_2 + x_2\sqrt{b})}{\sqrt{b}(z_0 + x_0\sqrt{c})} = \frac{\sqrt{c}(1 + \sqrt{b})}{\sqrt{b}(\pm 1 + \sqrt{c})} = \frac{1 + \sqrt{b}}{\sqrt{b}} \cdot \frac{\sqrt{c}}{\sqrt{c} \pm 1} > 1.$$

This and $c \geq r^2 + 2r$ give

$$0 < \log \gamma \leq \log \left(1 + \frac{1}{\sqrt{b}} \right) + \log \left(1 + \frac{1}{\sqrt{c} - 1} \right) < \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c} - 1} < \frac{2}{\sqrt{b}}.$$

It implies

$$|\Lambda - \log \gamma| < \left| \frac{1}{4c} - \frac{2}{\sqrt{b}} \right| < \frac{2}{\sqrt{b}}.$$

This and (25) prove the lemma in the case $\lambda = 0$.

Case II: If $\lambda = 1$ ($= \lambda_1$), then the solution (l, m) is of Type II, with $x_0 = r, x_2 = 1, z_0 = \lambda_1 t = t, y_2 = \lambda_2, \lambda_2 \in \{-1, 1\}$. We get

$$\gamma = \frac{\sqrt{c}(\lambda_2 + \sqrt{b})}{\sqrt{b}(t + r\sqrt{c})}.$$

As

$$\alpha^\lambda \gamma - 1 = \frac{\sqrt{c}(\lambda_2 + \sqrt{b})(r + \sqrt{b})}{\sqrt{b}(t + r\sqrt{c})} - 1 = \frac{\lambda_2 \sqrt{c}(r + \sqrt{b}) - \frac{\sqrt{b}}{t + \sqrt{bc}}}{\sqrt{b}(t + r\sqrt{c})},$$

we have

$$|\alpha^\lambda \gamma - 1| < \frac{\sqrt{c}(r + \sqrt{b}) + 0.01}{\sqrt{b}(t + r\sqrt{c})}.$$

As $\sqrt{c}(r + \sqrt{b}) - (t + r\sqrt{c}) = \sqrt{bc} - t \leq 0$, we see that

$$|\alpha^\lambda \gamma - 1| < \frac{1.01}{\sqrt{b}} < 0.011.$$

It results that

$$|\log(\alpha^\lambda \gamma)| = |\log(1 + (\alpha^\lambda \gamma - 1))| < \frac{1.02}{\sqrt{b}}.$$

Combining the above inequality and (25), we obtain

$$|(l - \lambda) \log \alpha - m \log \beta| < \frac{1}{4c} + \frac{1.02}{\sqrt{b}} < \frac{1.03}{\sqrt{b}}.$$

Case III: If $\lambda = -1$ ($= \lambda_1$), then the solution (l, m) is of Type II, with $x_0 = r, x_2 = 1, z_0 = -t, y_2 = \lambda_2, \lambda_2 \in \{-1, 1\}$. We have

$$\gamma = \frac{\sqrt{c}(\lambda_2 + \sqrt{b})}{\sqrt{b}(-t + r\sqrt{c})},$$

and so

$$\alpha^\lambda \gamma = \alpha^{-1} \gamma = \frac{\sqrt{c}(\lambda_2 + \sqrt{b})(t + r\sqrt{c})}{\sqrt{b}(c - 1)(r + \sqrt{b})}.$$

We get

$$\alpha^\lambda \gamma - 1 = \frac{t\sqrt{bc} - bc + \sqrt{b}(r + \sqrt{b}) + \lambda_2 \sqrt{c}(t + r\sqrt{c})}{\sqrt{b}(c - 1)(r + \sqrt{b})}.$$

With $t\sqrt{bc} - bc = \sqrt{bc}(t - \sqrt{bc}) = \frac{\sqrt{bc}}{t + \sqrt{bc}} < 1/2$, we obtain

$$|\alpha^\lambda \gamma - 1| < \frac{1}{2\sqrt{b}(c - 1)(r + \sqrt{b})} + \frac{1}{c - 1} + \frac{\sqrt{c}(t + r\sqrt{c})}{\sqrt{b}(c - 1)(r + \sqrt{b})}.$$

From

$$\frac{\sqrt{c}(t + r\sqrt{c})}{(c - 1)(r + \sqrt{b})} - 1 = \frac{r + \sqrt{b} + \frac{\sqrt{c}}{t + \sqrt{bc}}}{(c - 1)(r + \sqrt{b})} < \frac{r + \sqrt{b} + 0.01}{(c - 1)(r + \sqrt{b})} < \frac{1.01}{c - 1} < 0.02,$$

we get

$$|\alpha^\lambda \gamma - 1| < \frac{1}{2\sqrt{b}(c - 1)(r + \sqrt{b})} + \frac{1}{c - 1} + \frac{1.02}{\sqrt{b}} < \frac{1.04}{\sqrt{b}}.$$

Hence, we deduce

$$|(l - \lambda) \log \alpha - m \log \beta| < \frac{1}{4c} + \frac{1.04}{\sqrt{b}} < \frac{1.05}{\sqrt{b}}.$$

This completes the proof of the lemma. \square

Put

$$(26) \quad \Delta = l - \lambda - km.$$

Lemma 5. *If $P_l = Q_m$ has a solution (l, m) with $m > 1$, then $\Delta \neq 0$.*

Proof. Assume that $\Delta = l - \lambda - km = 0$. From (19), by induction one gets

$$(27) \quad P_0 = x_2, \quad P_k = x_2 T_k + y_2 U_k, \quad P_{mk+2k} = 2T_k P_{mk+k} - P_{mk}.$$

Case I: $\lambda = 0$. This is Type I with $x_0 = x_2 = 1, z_0 = \pm 1, y_2 = 1$. We

have $l = km$. By (27) and (20), we obtain

$$(28) \quad P_0 = 1, \quad P_k = T_k + U_k, \quad P_{mk+2k} = 2T_k P_{mk+k} - P_{mk}$$

and

$$(29) \quad Q_0 = 1, \quad Q_1 = s \pm 1, \quad Q_{m+2} = 2sQ_{m+1} - Q_m.$$

Recall that $s = s_k^{(\pm)} = T_k \pm U_k$.

When $s = s_k^{(-)}$, from $P_k > Q_1$ and $2T_k > 2s$, we have $P_{km} > Q_m$, for $m \geq 1$.

When $s = s_k^{(+)}$ and $Q_1 = s + 1$, by $P_k < Q_1$ and $2T_k < 2s$, we obtain $P_{km} < Q_m$, for $m \geq 1$.

When $s = s_k^{(+)}$ and $Q_1 = s - 1$, we get $P_k < Q_1$. From $2T_k < 2s$ and

$$P_{2k} = 2T_k(T_k + U_k) - 1 < 2(T_k + U_k)(T_k + U_k - 1) - 1 = Q_2$$

we have $P_{mk} < Q_m$, for $m \geq 2$.

Thus, we obtain $P_{km} \neq Q_m$ in Type I. This contradicts the fact that $l = km$.

Case II: $\lambda = 1$. We are in Type II with $\lambda_1 = 1, x_0 = r, x_2 = 1, z_0 = t, y_2 = \lambda_2 = \pm 1$. If $\Delta = 0$, then $l = km + 1$. By (28) and (20), we have

$$(30) \quad P_0 = 1, \quad P_k = T_k \pm U_k, \quad P_{mk+2k} = 2T_k P_{mk+k} - P_{mk}$$

and

$$(31) \quad Q_0 = r, \quad Q_1 = rs + t, \quad Q_{m+2} = 2sQ_{m+1} - Q_m.$$

If $m = 0$, then $l = 1$. $P_1 = rx_2 + y_2 = r \pm 1 \neq r = Q_0$. If $m = 1$, then $l = k + 1$. $P_{k+1} = r(T_k \pm U_k) \pm T_k + bU_k$. $Q_1 = rs + t = rs_k^{(\pm)} + t_k^{(\pm)} = r(T_k \pm U_k) + (\pm T_k + bU_k)$.

When $s = s_k^{(\lambda_2)}$, then $P_{k+1} = Q_1$. But by induction $2T_k \neq 2s$ provides $P_{km+1} \neq Q_m$, for $m \geq 2$.

When $s = s_k^{(+)}$ and $\lambda_2 = -1$. $P_{k+1} < Q_1$ and $2T_k < s$ imply $P_{km+1} < Q_m$.

When $s = s_k^{(-)}$ and $\lambda_2 = 1$. $P_{k+1} < Q_1$ and $2T_k > s$ imply $P_{km+1} > Q_m$.

Therefore, $P_{km+1} \neq Q_m$. It contradicts our assumption $l = km + 1$.

Case III: $\lambda = -1$. It is of Type II with $\lambda_1 = -1$, $x_0 = r$, $x_2 = 1$, $z_0 = -t$, $y_2 = \lambda_2 = \pm 1$. The proof is similar to that in Case II. \square

Lemma 6. *If $P_l = Q_m$ has a solution (l, m) with $m \geq 1$, then for $r \geq 1000$, we have*

$$m > 0.98|\Delta|\sqrt{b} \cdot \log \alpha.$$

Proof. From Lemma 4, we have $|(l - \lambda) \log \alpha - m \log \beta| < \frac{2}{\sqrt{b}}$ and then

$$\left| \frac{l - \lambda}{m} - \frac{\log \beta}{\log \alpha} \right| < \frac{2}{m\sqrt{b} \cdot \log \alpha}.$$

Thus, we have

$$(32) \quad \frac{|\Delta|}{m} = \left| \frac{l - \lambda - km}{m} \right| < \left| \frac{\log \beta}{\log \alpha} - k \right| + \frac{2}{m\sqrt{b} \cdot \log \alpha}.$$

Moreover, we get

$$(33) \quad \left| \frac{\log \beta}{\log \alpha} - k \right| = \left| \frac{\log \beta - \log \alpha^k}{\log \alpha} \right| = \left| \frac{\log(\beta/\alpha^k)}{\log \alpha} \right| = \left| \frac{\log(1 + (\beta - \alpha^k)/\alpha^k)}{\log \alpha} \right|.$$

As $\alpha^k = (r + \sqrt{b})^k = T_k + U_k\sqrt{b}$ and $\beta = s + \sqrt{c}$, with $s = s_k^{(\pm)} = T_k \pm U_k$, we have

$$\begin{aligned} \left| \frac{\beta - \alpha^k}{\alpha^k} \right| &= \left| \frac{s + \sqrt{c} - (r + \sqrt{b})^k}{(r + \sqrt{b})^k} \right| = \left| \frac{2s + \frac{1}{s+\sqrt{c}} - (2T_k + \frac{1}{T_k+U_k\sqrt{b}})}{T_k + U_k\sqrt{b}} \right| \\ &= \left| \frac{\pm 2U_k + \frac{1}{s+\sqrt{c}} - \frac{1}{T_k+U_k\sqrt{b}}}{T_k + U_k\sqrt{b}} \right| < \frac{2U_k + 0.01}{T_k + U_k\sqrt{b}} < \frac{2U_k + 0.01}{2U_k\sqrt{b}} < \frac{1.001}{\sqrt{b}}. \end{aligned}$$

We deduce that

$$(34) \quad \left| \log(1 + (\beta - \alpha^k)/\alpha^k) \right| < 1.01 \left| \frac{\beta - \alpha^k}{\alpha^k} \right| < \frac{1.012}{\sqrt{b}}.$$

Combining this, (32) and (33), we obtain

$$\frac{|\Delta|}{m} < \frac{1.02}{\sqrt{b} \cdot \log \alpha} + \frac{2}{m\sqrt{b} \cdot \log \alpha} = \frac{1.012 + \frac{2}{m}}{\sqrt{b} \cdot \log \alpha}.$$

Therefore, it results

$$1.012m + 2 > |\Delta|\sqrt{b} \cdot \log \alpha.$$

This implies

$$m > 0.98|\Delta|\sqrt{b} \cdot \log \alpha,$$

and the proof is completed. \square

Moreover, we have the following result.

Lemma 7. *If $P_l = Q_m$ has a solution (l, m) , then $m \equiv 0, \pm 1 \pmod{r}$.*

Proof. By induction, we have

$$P_{2l} \equiv x_2 \pmod{r}, \quad P_{2l+1} \equiv y_2 \pmod{r}.$$

From $t^2 - bs^2 = 1 - b$, $s = s_k^{(\pm)} = T_k + \lambda_3 U_k$ and the sequences of $\{T_k\}$, $\{U_k\}$ given by (3) and (4), we have $s \equiv \pm 1 \pmod{r}$. Let $s \equiv \pm \lambda_3 \pmod{r}$, $\lambda_3 \in \{-1, 1\}$. From (20), we get

$$Q_m \equiv \lambda_3^{m-1}(\lambda_3 x_0 + z_0 m) \pmod{r}.$$

We will consider two cases.

Type I: with $l \equiv m \equiv 0 \pmod{2}$ and $x_0 = 1, z_0 = \pm 1$. Then, we have

$$P_l \equiv x_2 = 1 \pmod{r}, \quad Q_m \equiv x_0 + \lambda_3 z_0 m = 1 \pm m \pmod{r},$$

and $m \equiv 0 \pmod{r}$.

Type II: with $m \equiv 1 \pmod{2}$ and $x_0 = r, z_0 = \pm t, y_2 = \pm 1$. Thus, we get

$$P_l \equiv \pm 1 \pmod{r}, \quad Q_m \equiv \lambda_3 r + z_0 m \equiv z_0 m \pmod{r}.$$

The fact that $t = \pm T_k + bU_k \equiv \pm T_k - U_k \equiv \pm s \pmod{r}$ helps to obtain $Q_m \equiv z_0 m \equiv \pm m \pmod{r}$. Therefore, we deduce $m \equiv \pm 1 \pmod{r}$. \square

4. LINEAR FORMS IN TWO LOGARITHMS

Now, we recall the following result due to Laurent (see [9], Corollary 2) on linear forms in two logarithms. For any non-zero algebraic number α of degree d over \mathbb{Q} , whose minimal polynomial over \mathbb{Z} is $a \prod_{j=1}^d (X - \alpha^{(j)})$, we denote by

$$h(\alpha) = \frac{1}{d} \left(\log |a| + \sum_{j=1}^d \log \max(1, |\alpha^{(j)}|) \right)$$

its absolute logarithmic height.

Lemma 8. *Let α_1 and α_2 be multiplicatively independent algebraic numbers and $\alpha_1, \alpha_2, \log \alpha_1, \log \alpha_2$ are real and positive, b_1 and $b_2 \in \mathbb{Z}$ and*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1.$$

Let $D := [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$, for $i = 1, 2$ let

$$h_i \geq \max \left\{ h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D} \right\}$$

and

$$b' \geq \frac{|b_1|}{D h_2} + \frac{|b_2|}{D h_1}.$$

If $|\Lambda| \neq 0$, then we have

$$\log |\Lambda| \geq -17.9 \cdot D^4 \left(\max \left\{ \log b' + 0.38, \frac{30}{D}, \frac{1}{2} \right\} \right)^2 h_1 h_2.$$

As stated in Section 1, we only need to consider the extensibility of the triples $\{1, b, c_1^{(+)}\}$ and $\{1, b, c_k^{(\pm)}\}$, for $2 \leq k \leq 7$. Assume that $r \geq 1000$ and $P_l = Q_m$ has a solution (l, m) with $l, m \geq 1$. We have

$$\Lambda = l \log \alpha - m \log \beta + \log \gamma.$$

In order to apply Lemma 8, we put $\Delta = l - \lambda - km$ and rewrite Λ into the form

$$(35) \quad \Lambda = \log (\alpha^{\Delta+\lambda} \gamma) - m \log \left(\frac{\beta}{\alpha^k} \right).$$

Hence, we take

$$D = 4, \quad b_1 = m, \quad b_2 = 1, \quad \alpha_1 = \frac{\beta}{\alpha^k}, \quad \alpha_2 = \alpha^{\Delta+\lambda} \gamma.$$

One can check $\alpha_1, \alpha_2, \log \alpha_1$ and $\log \alpha_2$ are real and positive. Otherwise we work on $\Lambda = m \log(\alpha^k/\beta) - \log(\alpha^{-\Delta-\lambda} \gamma^{-1})$.

Since $\alpha = r + \sqrt{b}$, $\beta = s + \sqrt{c}$, then $\alpha_1 = \frac{(r+\sqrt{b})^k}{s+\sqrt{c}} = \frac{T_k+U_k\sqrt{b}}{s+\sqrt{c}}$ is a zero of the polynomial

$$X^4 + 4sT_kX^3 - 4(T_k^2 + s^2 + 1)X^2 + 4sT_kX + 1.$$

The absolute values of its conjugates whose greater than 1 are $\alpha^k \beta$ and

$$\begin{cases} \beta/\alpha^k, & \text{if } s = s_k^{(+)}, \\ \alpha^k/\beta, & \text{if } s = s_k^{(-)}. \end{cases}$$

Hence

$$h(\alpha_1) = \frac{1}{4} \max \{ (\log(\alpha^k \beta) \pm \log(\alpha^k/\beta)) \} = \frac{k}{2} \log \alpha \quad \text{or} \quad \frac{1}{2} \log \beta.$$

Also, it is easy to see that $h(\alpha^{\Delta+\lambda}) = \frac{1}{2} |\Delta + \lambda| \cdot \log \alpha$ and

$$h(\gamma) = h \left(\frac{\sqrt{c}(y_2 + x_2\sqrt{b})}{\sqrt{b}(z_0 + x_0\sqrt{c})} \right) \leq h \left(\frac{y_2 + x_2\sqrt{b}}{\sqrt{b}} \right) + h \left(\frac{z_0 + x_0\sqrt{c}}{\sqrt{c}} \right)$$

$$\leq \frac{1}{2} \log(b + \sqrt{b}) + \frac{1}{2} \log(rc + t\sqrt{c}) < \frac{1}{2} \log(4rbc) < \log \alpha + \frac{1}{2} \log \beta.$$

Thus, we have

$$h(\alpha_2) = h(\alpha^{\Delta+\lambda}\gamma) \leq h(\alpha^{\Delta+\lambda}) + h(\gamma) \leq \frac{1}{2}(|\Delta + \lambda| + 2) \log \alpha + \frac{1}{2} \log \beta.$$

As $r \geq 1000$, we obtain $\frac{|\alpha^k - \beta|}{\alpha^k} < \frac{1}{\sqrt{b}} < 0.001$. This helps to get $|\log \alpha^k - \log \beta| < 0.002$ and then

$$h_1 = \frac{k}{2} \log \alpha + 0.01 > h(\alpha_1), \quad h_2 = \frac{1}{2}(|\Delta + \lambda| + 2 + k) \log \alpha + 0.01 < h(\alpha_2).$$

Moreover, we have $\frac{|b_2|}{Dh_1} = \frac{1}{2k \log \alpha + 0.04} < 0.07$. So we put

$$(36) \quad b' = \frac{m}{2(|\Delta + \lambda| + 2 + k) \log \alpha + 0.04} + 0.07.$$

We will bound b' . If $\log b' + 0.38 \leq 30/D = 7.5$, then

$$b' < 1237.$$

Otherwise, by Lemma 8 we obtain

$$(37) \quad \log |\Lambda| \geq -17.9 \cdot 4^4 (\log b' + 0.14)^2 h_1 h_2.$$

On the other hand, from Lemma 3 we get $\log |\Lambda| < -2m \log \beta$. Thus we have

$$m \log \beta < 17.9 \cdot 128 (\log b' + 0.38)^2 h_1 h_2.$$

Since $\log \beta > \log \alpha^k - 0.002 > 2h_1 - 0.03$, then

$$m < 1.01 \cdot 17.9 \cdot 64 (\log b' + 0.38)^2 h_2.$$

It follows that

$$b' - 0.07 = \frac{m}{4h_2} < 289.27 (\log b' + 0.38)^2.$$

We calculate that $b' < 33791$ (> 1237). Therefore, we get the following.

Lemma 9. *For a triple $\{1, b, c_k^{(\pm)}\}$, ($1 \leq k \leq 7$), if $P_l = Q_m$ has a solution (l, m) with $m \geq 1$ and $r \geq 1000$, then we have*

$$m < 67582(|\Delta + \lambda| + 2 + k) \log \alpha + 1352.$$

The following result comes from Lemmas 6 and 9.

Proposition 1. *For a triple $\{1, b, c_k^{(\pm)}\}$, ($1 \leq k \leq 7$), if $r > 68962(k + 4) + 115$, then the equation $P_l = Q_m$ has no solution (l, m) satisfying $m > 1$.*

Proof. Assume that $r \geq 1000$. Since $\Delta \neq 0$, then $|\Delta| \geq 1$. By Lemma 6 and Lemma 9, we have

$$0.98|\Delta|\sqrt{b}\log\alpha < 67582(|\Delta + \lambda| + 2 + k)\log\alpha + 1352.$$

This implies

$$\begin{aligned} r - 1 < \sqrt{b} &< \frac{67582(|\Delta + \lambda| + 2 + k)}{0.98|\Delta|} + \frac{1866.1}{0.98|\Delta|\log\alpha} \\ &< 68962(k + 4) + 114. \end{aligned}$$

Therefore, this completes the proof of Proposition 1. \square

5. PROOF OF THEOREM 1

In this section, we will use another theorem for the lower bounds of linear forms in logarithms which differs from that in above section and the Baker-Davenport reduction method to deal with the remaining cases. We recall the following result is due to Matveev [10].

Lemma 10. *Denote by $\alpha_1, \dots, \alpha_j$ algebraic numbers, not 0 or 1, by $\log \alpha_1, \dots, \log \alpha_j$ determinations of their logarithms, by D the degree over \mathbb{Q} of the number field $\mathbb{K} = \mathbb{Q}(\alpha_1, \dots, \alpha_j)$, and by b_1, \dots, b_j rational integers. Define $B = \max\{|b_1|, \dots, |b_j|\}$, and $A_i = \max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\}$ ($1 \leq i \leq j$), where $h(\alpha)$ denotes the absolute logarithmic Weil height of α . Assume that the number*

$$\Lambda = b_1 \log \alpha_1 + \dots + b_n \log \alpha_j$$

does not vanish; then

$$|\Lambda| \geq \exp\{-C(j, \varkappa)D^2 A_1 \cdots A_j \log(eD) \log(eB)\},$$

where $\varkappa = 1$ if $\mathbb{K} \subset \mathbb{R}$ and $\varkappa = 2$ otherwise and

$$C(l, \varkappa) = \min \left\{ \frac{1}{\varkappa} \left(\frac{1}{2} e j \right)^{\varkappa} 30^{j+3} j^{3.5}, 2^{6j+20} \right\}.$$

Now, we apply the above lemma with $j = 3$ and $\varkappa = 1$ for

$$\Lambda = l \log \alpha - m \log \beta + \log \gamma.$$

Here we take

$$D = 4, \quad b_1 = l, \quad b_2 = -m, \quad b_3 = 1, \quad \alpha_1 = \alpha, \quad \alpha_2 = \beta, \quad \alpha_3 = \frac{\sqrt{c}(y_2 + x_2\sqrt{b})}{\sqrt{b}(z_0 + x_0\sqrt{c})}.$$

From the computations done in the previous section, we put

$$h(\alpha_1) = \frac{1}{2} \log \alpha, \quad h(\alpha_2) = \frac{1}{2} \log \beta.$$

We see that α_3 is a zero of the polynomial

$$b^2(c-1)^2 X^4 - 4b^2 c(c-1)x_0 x_2 X^3 - 2bc((b-1)(c-1) - 2b(c-1)x_2^2 - 2c(b-1)x_0^2) X^2$$

$$-4bc^2(b-1)x_0x_2X + c^2(b-1)^2.$$

This implies

$$\begin{aligned} h(\alpha_3) &\leq \frac{1}{4} \left(\log(b^2(c-1)^2) + 4 \log \frac{\max\{|\sqrt{c}(y_2 \pm x_2\sqrt{b})|\}}{\min\{|\sqrt{b}(z_0 \pm x_0\sqrt{c})|\}} \right) \\ &= \frac{1}{4} \left(\log(b^2(c-1)^2) + 4 \log \frac{\sqrt{c}(1+\sqrt{b})}{\sqrt{b}(-t+r\sqrt{c})} \right) \\ &\leq \frac{1}{4} \left(\log \left(\frac{2^4 c^4 (1+\sqrt{b})^4}{(c-1)^2} \right) \right) \leq \log(2bc). \end{aligned}$$

Therefore, we take

$$A_1 = 2 \log \alpha, \quad A_2 = 2 \log \beta, \quad A_3 = 4 \log(2bc).$$

Using Matveev's result we have

$$(38) \quad \log |\Lambda| > -1.3901 \cdot 10^{11} \cdot 16 \cdot \log \alpha \cdot \log \beta \cdot \log(2bc) \cdot \log(4e) \cdot \log(2el).$$

By Lemma 3, we know that $\log |\Lambda| < -2m \log \beta$. It is easy to show that $m \log \beta > 0.5l \log \alpha$. Combining the two bounds for $\log |\Lambda|$, we get

$$\frac{l}{\log(2el)} < 5.4 \cdot 10^{12} \cdot \log \beta \cdot \log(2bc) < 5.4 \cdot 10^{12} \cdot \log^2(2c^2).$$

As $c = c_k^{(\pm)} \leq c_k^{(+)}$, $k \leq 7$ and $r \leq 68962(k+4) + 115$, we have $c < (2r)^{14} < 3.44 \cdot 10^{86}$. The above inequality gives $l < 2.2 \cdot 10^{17}$.

In order to deal with the remaining cases, we will use a Diophantine approximation algorithm called the Baker-Davenport reduction method. The following lemma is a slight modification of the original version of Baker-Davenport reduction method. (See [6, Lemma 5a]).

Lemma 11. *Assume that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$ and let*

$$\eta = \|\mu q\| - M \cdot \|\kappa q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < l\kappa - m + \mu < AB^{-l}$$

in integers l and m with

$$\frac{\log(Aq/\eta)}{\log B} \leq l \leq M.$$

We apply Lemma 11 to Λ given by (21) with

$$\kappa = \frac{\log \alpha}{\log \beta}, \quad \mu = \frac{\log \gamma}{\log \beta}, \quad A = 1, \quad B = \alpha, \quad \text{and} \quad M = 2.2 \cdot 10^{17}.$$

The program was developed in PARI/GP running with 200 digits precision. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find the one that satisfies the conditions. In 11 hours, all the computations were done (using an Intel i7 4960HQ CPU). In all cases, we obtained $l \leq 42$. From Lemma 7, we have $m \geq r - 1$. By $m \leq l$, we get $r \leq 1 + m < l + 1 \leq 43$. The second running provided $l \leq 9$. We checked all cases and found no solution to $P_l = Q_m$, for $m \geq 2$. Then, we have.

Proposition 2. *For a triple $\{1, b, c_k^{(\pm)}\}$, ($1 \leq k \leq 7$), if $r \leq 68962(k + 4) + 115$, then equation $P_l = Q_m$ has no solution (l, m) satisfying $m > 1$.*

Combining Proposition 1 and Proposition 2, we deduce that if $P_l = Q_m$ has a positive integer solution (l, m) , then $m \leq 1$. In fact, a solution comes from $m = 1, l = k + 1, z_0 = t$, which implies that $x = T_{k+1} \pm U_{k+1}, y = \pm T_{k+1} + bU_{k+1}$. Thus, we get

$$d = x^2 - 1 = (T_{k+1} \pm U_{k+1})^2 - 1 = \pm 2T_{k+1}U_{k+1} + r^2U_{k+1}^2 = c_{k+1}^{(\pm)}.$$

This completes the proof of our Theorem.

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